The Vasicek Interest Rate Process
Modeling Random Bond Price

Gary Schurman, MBE, CFA

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We will define a bond forward contract to be a binding agreement between two parties entered into at time zero to either buy or sell a zero coupon bond at time s that pays face value at time t where $0 < s < t$. The contractual price to be paid for the bond at time s is agreed to at time zero. Whereas the forward bond price is not a random variable from the standpoint of time zero the actual bond price at time s is a random variable.

In this white paper we will develop the mathematics to price a bond forward assuming that interest rates evolve per the Vasicek interest rate process. We will then use the mathematics to develop an equation for random bond price at time $s > 0$. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the price to be paid at time s for a bond that matures at time t. Our go-forward model assumptions are...

Description	Symbol	Value
Bond face value (in dollars)	FV	1,000
Bond purchase time (in years)	s	3.00
Bond maturity time (in years)	t	7.00
Short rate at time zero	r_0	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: From the perspective of time zero, what is the bond forward price at the end of year three?

Question 2: From the perspective of time zero, what is the probability that random bond price at the end of year three will be greater than $800.

Bond Price Equation

We will define the variable $r_{n,s}$ to be the random short rate at time s given the known short rate at time n, and the variable $M(r_{n,s})$ to be the mean of the random short rate $r_{n,s}$. If the variable r_n is the known short rate at time n and the variable r_∞ is the short rate long-term mean, then the equation for the mean of the random short rate is... [1]

$$M(r_{n,s}) = r_\infty + \mathrm{Exp}\left\{ -\lambda\,(s-n)\right\}\left(r_n - r_\infty\right) \tag{1}$$

We will define the variable $V(r_{n,s})$ to be the variance of the random short rate $r_{n,s}$. The equation for the variance of the random short rate is... [1]

$$V(r_{n,s}) = \frac{1}{2}\,\sigma^2\left(1 - \mathrm{Exp}\left\{ -2\,\lambda\,(s-n)\right\}\right)\lambda^{-1} \tag{2}$$

Using Equations (1) and (2) above the equation for the random short rate at time s given the known short rate at time n is...

$$r_{n,s} = M(r_{n,s}) + \sqrt{V(r_{n,s})}\, Z \quad \text{...where...} \quad Z \sim N\Big[0,1\Big] \tag{3}$$

We defined the function $B(s,t)$ as follows... [4]

$$B(s,t) = \left(1 - \text{Exp}\left\{ -\lambda\,(t-s)\right\}\right)\lambda^{-1} \tag{4}$$

We defined the function $A(s,t)$ as follows... [4]

$$A(s,t) = \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{4\,\lambda}\,B^2(s,t) \tag{5}$$

We defined the variable $P(s,t)$ to be the price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time s. Using Equations (1), (4) and (5) above the equation for zero coupon bond price at time s is... [4]

$$P(s,t) = \text{Exp}\left\{A(s,t) - B(s,t)\,r_{s,s}\right\} \quad \text{...where...} \quad r_{s,s} \text{ is the known short rate at time } s \text{ (i.e. not random)} \tag{6}$$

Using Equation (6) above the equation for the log of zero coupon bond price is... [4]

$$\ln P(s,t) = A(s,t) - B(s,t)\,r_{s,s} \quad \text{...where...} \quad r_{s,s} \text{ is the known short rate at time } s \text{ (i.e. not random)} \tag{7}$$

Note that in bond price Equations (6) and (7) above the variable $P(s,t)$ is a non-random variable because the short rate $r_{s,s}$ is a non-random variable.

Forward Bond Price Equation

We will define the variable I_t to be the dollar value of an investment portfolio at time t. Imagine that we are currently standing at time n and want to invest I_n at the risk-free rate over the time interval $[n,t]$. We will consider the following two investment options...

Option A - Invest I_n at the risk-free rate over the time interval $[n,t]$. We do this by buying zero coupon bonds $P(n,t)$ as defined by Equation (6) above. The equation for portfolio value at time t is...

$$I_t = I_n \times \frac{1}{P(n,t)} \quad \text{...where...} \quad n < t \tag{8}$$

Option B - Invest I_n at the risk-free rate over the shorter time interval $[n,s]$. We do this by buying zero coupon bonds $P(n,s)$ as defined by Equation (6) above. The equation for portfolio value at time s is...

$$I_s = I_n \times \frac{1}{P(n,s)} \quad \text{...where...} \quad n < s \tag{9}$$

When the bond portfolio in Equation (9) above matures at time s we invest I_s at the risk-free rate over the time interval $[s,t]$. To lock in the interest rate over the time interval $[s,t]$ we buy the bond $P(s,t)$ at the forward price $F(n,s,t)$. The equation for portfolio value at time t is...

$$I_t = I_s \times \frac{1}{F(n,s,t)} \quad \text{...where...} \quad s < t \tag{10}$$

Using Equations (8), (9) and (10) above, to prevent arbitrage the following equation must hold...

$$I_n \times \frac{1}{P(n,t)} = I_n \times \frac{1}{P(n,s)} \times \frac{1}{F(n,s,t)} \tag{11}$$

Using Equation (11) above and solving for forward bond price...

$$F(n,s,t) = \frac{P(n,t)}{P(n,s)} \tag{12}$$

Using Equation (12) above the equation for the log of forward bond price is...

$$\ln F(n,s,t) = \ln\left(\frac{P(n,t)}{P(n,s)}\right) = \ln P(n,t) - \ln P(n,s) \tag{13}$$

Note that $F(n,s,t)$ is price agreed to at time n to purchase the bond $P(s,t)$ at time s. Given that bond prices $P(n,s)$ and $P(n,t)$ are both known at time n (i.e. are not random) then the forward price $F(n,s,t)$ in Equations (12) and (13) above are not a random variables.

Random Bond Price Equation

Using Equations (12) and (13) above the equation for the expected price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time n (i.e. the short rate at time s is a random variable) is...

$$\mathbb{E}\left[P(n,s,t)\right] = F(n,s,t) = \frac{P(n,t)}{P(n,s)} \text{ ...such that... } \mathbb{E}\left[\ln P(n,s,t)\right] = \ln F(n,s,t) = \ln\left(\frac{P(n,t)}{P(n,s)}\right) \tag{14}$$

We will define the variables m and v to be the random return mean and variance, respectively, of random bond price. Note that we can rewrite Equation (14) above as... [3]

$$\mathbb{E}\left[P(n,s,t)\right] = \text{Exp}\left\{-m + \frac{1}{2}v\right\} \text{ ...such that... } \ln F(n,s,t) = -m + \frac{1}{2}v \tag{15}$$

If we equate Equations (14) and (15) above then we get the following equations...

$$\text{if... } \mathbb{E}\left[\ln P(s,t)\right] = \ln F(n,s,t) \text{ ...then... } \ln\left(\frac{P(n,t)}{P(n,s)}\right) = -m + \frac{1}{2}v \tag{16}$$

The variable v in Equation (16) above is determined via Appendix Equation (32) below. The equations for the variables m and v are therefore...

$$m = -\ln\left(\frac{P(n,t)}{P(n,s)}\right) + \frac{1}{2}v \text{ ...where... } v = B(s,t)^2\, V(r_{n,s}) \tag{17}$$

Simulating Random Bond Price

We defined the variable $P(n,s,t)$ to be the random dollar price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time n (i.e. the short rate at time s is a random variable). Using Equations (15) and (17) above the equation to simulate the random bond price is...

$$P(n,s,t) = \text{Exp}\left\{-m + \sqrt{v}\,Z\right\} \text{ ...where... } Z \sim N\left[0,1\right] \tag{18}$$

Using Equation (18) above and solving for the random variable Z...

$$Z = \frac{\ln P(n,s,t) + m}{\sqrt{v}} \tag{19}$$

We will define the function $CNDF(Z)$ to be the cumulative normal distribution function of the normally-distributed random variable Z with mean zero and variance one. Using Equation (19) above the probability that the dollar bond price $P(n,s,t)$ at time s is less than some threshold value θ is...

$$\text{Prob}\left[P(n,s,t < \theta\right] = CNDF(Z) = CNDF\left(\frac{\ln(\theta) + m}{\sqrt{v}}\right) \tag{20}$$

Answers To Our Hypothetical Problem

Using Equation (4) above the equation for bond pricing parameter $B(s,t)$ is...

$$B(0,3) = \left(1 - \text{Exp}\left\{-0.35 \times (3-0)\right\}\right) \times 0.35^{-1} = 1.8573$$

$$B(0,7) = \left(1 - \text{Exp}\left\{-0.35 \times (7-0)\right\}\right) \times 0.35^{-1} = 2.6106$$

$$B(3,7) = \left(1 - \text{Exp}\left\{-0.35 \times (7-3)\right\}\right) \times 0.35^{-1} = 2.1526 \tag{21}$$

Using Equations (5) and (21) above the equation for bond pricing parameter $A(s,t)$ is...

$$A(0,3)=\left(0.09-\frac{0.03^2}{2\times 0.35^2}\right)\times\left(1.8573-(3-0)\right)-\frac{0.03^2}{4\times 0.35}\times 1.8573^2=-0.1009$$
$$A(0,7)=\left(0.09-\frac{0.03^2}{2\times 0.35^2}\right)\times\left(2.6106-(7-0)\right)-\frac{0.03^2}{4\times 0.35}\times 2.6106^2=-0.3833 \tag{22}$$

Using Equation (2) above the variance of the short rate over the time interval $[0,3]$ is...

$$V(r_{0,3})=\frac{1}{2}\times 0.03^2\times\left(1-\text{Exp}\left\{-2\times 0.35\times(3-0)\right\}\right)\times 0.35^{-1}=0.001128 \tag{23}$$

Using Equations (6), (21) and (22) above the equation for bond price is...

$$P(0,3)=1,000\times\text{Exp}\left\{-0.1009-1.8573\times 0.04\right\}=839.33$$
$$P(0,7)=1,000\times\text{Exp}\left\{-0.3833-2.6106\times 0.04\right\}=614.02 \tag{24}$$

Using Equations (17), (21), (23) and (24) above

$$m=-\ln\left(\frac{614.02}{839.33}\right)+\frac{1}{2}\times 0.00523=0.31519\ \text{...where...}\ v=2.1526^2\times 0.001128=0.00523 \tag{25}$$

Question 1: From the perspective of time zero, what is the bond forward price at the end of year three?

Using Equations (12) and (24) above the answer to the question is...

$$F(0,3,7)=1,000\times\frac{614.02}{839.33}=731.56 \tag{26}$$

Question 2: From the perspective of time zero, what is the probability that random bond price at the end of year three will be greater than $800.

Using Equations (20) and (25) above the answer to the question is...

$$\text{Prob}\left[1,000\times P(n,s,t)>800\right]=1-CNDF\left(\frac{\ln(800/1000)+0.31519}{\sqrt{0.00523}}\right)=0.1015 \tag{27}$$

Appendix

A. Note the following expectations applicable to the normally-distributed random variable Z with mean zero and variance one...

$$\text{if...}\ Z\sim N\left[0,1\right]\ \text{...then...}\ \mathbb{E}\left[Z\right]=0\ \text{...and...}\ \mathbb{E}\left[Z^2\right]=1 \tag{28}$$

B. Using Equations (3) and (7) above the equation for the log of random bond price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time $n<s$. The equation for the log of random bond price is...

$$\begin{aligned}\ln P(n,s,t)&=A(s,t)-B(s,t)\,r_{n,s}\\&=A(s,t)-B(s,t)\left(M(r_{n,s})+\sqrt{V(r_{n,s})}\,Z\right)\\&=A(s,t)-B(s,t)\,M(r_{n,s})-B(s,t)\sqrt{V(r_{n,s})}\,Z\end{aligned} \tag{29}$$

C. Using the definitions in Equation (28) above the first moment of distribution of the log of random bond price as defined by Equation (29) above is...

$$\begin{aligned}\mathbb{E}\left[\ln P(n,s,t)\right]&=\mathbb{E}\left[A(s,t)-B(s,t)\,M(r_{n,s})-B(s,t)\sqrt{V(r_{n,s})}\,Z\right]\\&=A(s,t)-B(s,t)\,M(r_{n,s})-B(s,t)\sqrt{V(r_{n,s})}\,\mathbb{E}\left[Z\right]\\&=A(s,t)-B(s,t)\,M(r_{n,s})\end{aligned} \tag{30}$$

D. Using the definitions in Equation (28) above the second moment of distribution of the log of random bond price as defined by Equation (29) above is...

$$\begin{aligned}
\mathbb{E}\Bigg[\bigg(\ln P(n,s,t)\bigg)^2\Bigg] &= \mathbb{E}\Bigg[A(s,t) - B(s,t)\,r_{n,s}\ \text{mean} - B(s,t)\,\sqrt{V(r_{n,s})}\,Z\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\ln P(n,s,t)\Bigg]^2 - 2\,A(s,t)\,B(s,t)\,\sqrt{V(r_{n,s})}\,Z + 2\,B(s,t)^2\,M(r_{n,s})\,\sqrt{V(r_{n,s})}\,Z \\
&+ B(s,t)^2\,V(r_{n,s})\,Z^2\Bigg] \\
&= \mathbb{E}\Bigg[\ln P(n,s,t)\Bigg]^2 - 2\,A(s,t)\,B(s,t)\,\sqrt{V(r_{n,s})}\,\mathbb{E}\Bigg[Z\Bigg] + 2\,B(s,t)^2\,M(r_{n,s})\,\sqrt{V(r_{n,s})}\,\mathbb{E}\Bigg[Z\Bigg] \\
&+ B(s,t)^2\,V(r_{n,s})\,\mathbb{E}\Bigg[z^2\Bigg] \\
&= \mathbb{E}\Bigg[\ln P(n,s,t)\Bigg]^2 + B(s,t)^2\,V(r_{n,s})
\end{aligned} \tag{31}$$

E. Using Equations (30) and (31) above the variance of the log of random bond price at time s of a zero coupon bond that pays one dollar at time t given the known short rate at time $n < s$ is...

$$v = \mathbb{E}\Bigg[\bigg(\ln P(n,s,t)\bigg)^2\Bigg] - \Bigg[\mathbb{E}\Bigg[\ln P(n,s,t)\Bigg]\Bigg]^2 = B(s,t)^2\,V(r_{n,s}) \tag{32}$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Discount Rate,* February, 2013.

[3] Gary Schurman, *The Vasicek Interest Rate Process - Zero Coupon Bond Price,* February, 2013.

[4] Gary Schurman, *The Vasicek Interest Rate Process - An Alternative Bond Price Equation,* December, 2021.